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ON SUBMERGED JETS*

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A non-similarity, axisymmetric, and in general twisted, jet of incompressible fluid, flowing from a spherical source into a submerged space, is considered. It is shown that the solution of this problem is not expressible by a series of integral inverse powers of the spherical radius R, but has an essentially singular point at $R=\infty$. The principal terms of the asymptotic expansion are given by four integrals of conservation, namely, momentum, flow rate, and two components of angular momentum, and not by three.

There is now a vast literature on the theory of laminar submerged jets, starting with the fundamental papers /1-3/, and including the special monographs /4, 5/. The interest stems from the fact that jet flows are a basic type of fluid and gas motion.

Landau /2/ interpreted his exact solution of the Navier-Stokes equations as the viscous fluid flow produced by a point singularity (the end of a thin tube), communicating finite momentum to the fluid at zero flow rate. This is a similarity solution in the velocity field, inversely proportional to the distance R from the jet source. According to Rumer's hypothesis /3/, a non-similarity solution with a finite flow rate can be constructed as an expansion in integral powers of $1/R$ with coefficients that depend on the spherical angle θ. This approach was used in all subsequent work, e.g., /6-10/; the Navier-Stokes equations were solved in /10/, and the boundary layer equations in /6-9/. It has been assumed that this expansion gives the asymptotic form of the solution remote from the actual jet source. Unfortunately, this assumption is incorrect.

1. We consider the axisymmetric stationary flow of an incompressible viscous fluid, at rest at infinity, caused by a given velocity field on a sphere, radius R_0, centre the origin of a spherical coordinate system R, θ, φ. The motion is described by the Navier-Stokes equations

$$(\mathbf{v}\nabla)\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

We shall show that this problem does not have a solution in the form of an expansion in integral powers of $1/R$. We will first take the case of an untwisted jet, when $v_\varphi \equiv 0$. Introducing the stream function ψ by the relations

$$v_R = \frac{1}{R^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{1}{R\sin\theta}\frac{\partial\psi}{\partial R} \tag{1.2}$$

and the independent variable $x=\cos\theta$, we will seek the solution in accordance with /3/ as the expansions

$$\psi = \nu R y_1(x) + \nu y_2(x) + O\left(\frac{1}{R}\right) \tag{1.3}$$

$$v_R = -\frac{\nu y_1'}{R} - \frac{\nu y_2'}{R^2} + O\left(\frac{1}{R^3}\right), \quad v_\theta = -\frac{\nu y_1}{R\sqrt{1-x^2}} + O\left(\frac{1}{R^3}\right)$$

$$\frac{p}{\rho} = \frac{\nu^2 g_1(x)}{R^2} + \frac{\nu^2 g_2(x)}{R^3} + O\left(\frac{1}{R^4}\right)$$

Substituting (1.3) into system (1.1) and equating coefficients of like powers of $1/R$, we obtain a recurrent system of equations, the first of which for y_1 is autonomous but non-linear, while the rest are linear. The function $y_1(x)$ is Landau's solution /2/

$$y_1(x) = 2\,\frac{1-x^2}{A-x} \tag{1.4}$$

where $A>1$ is a constant, connected with the total flux of momentum through a sphere of arbitrary radius R

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$$J=2\pi\rho R^2\int_0^\pi\left\{\left(v_R^2+\frac{p}{\rho}-2\nu\frac{\partial v_R}{\partial R}\right)\cos\theta-\left[v_Rv_\theta-\nu\left(\frac{\partial v_\theta}{\partial R}-\frac{v_\theta}{R}+\frac{1}{R}\frac{\partial v_R}{\partial\theta}\right)\right]\sin\theta\right\}\sin\theta\,d\theta \tag{1.5}$$

by the relation

$$J=16\pi\rho\nu^2A\left[1+\frac{4}{3(A^2-1)}-\frac{A}{2}\ln\frac{A+1}{A-1}\right] \tag{1.6}$$

As the parameter A increases from 1 to ∞, J falls montonically from ∞ to 0. For y_2' we obtain the equation

$$Ly_2'=c_1,\quad L=\frac{d}{dx}(1-x^2)\frac{d}{dx}-2\frac{1-x^2}{A-x}\frac{d}{dx}+6\frac{A^2-1}{(A-x)^2} \tag{1.7}$$

The constant c_1 is connected with the fluid flow rate Q through a spherical surface of any radius R

$$Q=2\pi\rho R^2\int_0^\pi v_R\sin\theta\,d\theta=-2\pi\rho\int_0^\pi \nu y_2'\sin\theta\,d\theta=-2\pi\mu\left[y_2(1)-y_2(-1)\right] \tag{1.8}$$

by the relation

$$c_1=-\frac{9(A^2-1)}{2\pi\rho(3A^2+1)}Q$$

The boundary conditions for (1.7) amount to requiring that y_2' be regular for $x=\pm1$. The non-trivial solution of the homogeneous Eq.(1.7) was found in /3/:

$$y_{20}'=c_2F_0(x),\ c_2=\mathrm{const} \tag{1.9}$$
$$F_0(x)=1-\frac{3(A^2-1)}{(A-x)^2}+\frac{2(A^2-1)^2}{A(A-x)^3}$$

which corresponds to zero flow rate. For the existence of a solution with non-zero flow rate, the constant c_1 must be orthogonal to the eigenfunction of the operator, adjoint to L in (1.7). Multiplication of Eq.(1.7) by $(A-x)^2$ leads to the selfadjoint equation

$$\frac{d}{dx}(1-x^2)(A-x)^2\frac{dy_2'}{dx}+6(A^2-1)y_2'=c_1(A-x)^2 \tag{1.10}$$

the condition for the solvability of which consists in the orthogonality of (1.9) and the right-hand side of (1.10)

$$c_1\int_{-1}^{1}(A-x)^2F_0\,dx=c_1\left[\frac{20}{3}-4A^2+\frac{2(A^2-1)}{A}\ln\frac{A+1}{A-1}\right]=0 \tag{1.11}$$

This last equation is not satisfied for any $A>1$, i.e., (1.7) is not, in general, solvable in the class of regular $y_2'(x)$. (Notice that (1.11) is satisfied for $A=\infty$ $(J=0)$, so that in this case expansion (1.3) is admissible.)

The solution of (1.7), quoted in /3/,

$$y_2'=F_0(x)\int_{-1}^{x}\frac{d\xi}{(1-\xi^2)(A-\xi)^2F_0^2(\xi)}\int_1^\xi(A-\eta)^2F_0(\eta)\,d\eta\,,$$

is not regular, since the integral with respect to ξ is logarithmically divergent at the point $\xi=-1$, not to mention the fact that, as may be seen by analysing (1.9), $F_0(x)$ has two zeros in the interval $(-1,1)$.

2. The fact that (1.7) is not solvable does not imply that the initial problem is unsolvable, but simply means that expansion (1.3) is unsuitable. Let us construct a more general expansion, assuming that the leading term $\mathbf{v}_1$ has, as in (1.3), the order $1/R$, as required by the law of conservation of momentum (1.5). We can then write

$$\mathbf{v}=\mathbf{v}_1+\mathbf{w},\quad \mathbf{w}=o(R^{-1}) \tag{2.1}$$

If $\mathbf{v}_1$ satisfies Eqs.(1.1), we obtain for the vector $\mathbf{w}$ the system

$$(\mathbf{w}\nabla)\mathbf{w}+(\mathbf{v}_1\nabla)\mathbf{w}+(\mathbf{w}\nabla)\mathbf{v}_1=-\rho^{-1}\nabla q+\nu\Delta\mathbf{w},\quad \mathrm{div}\,\mathbf{w}=0 \tag{2.2}$$

In jet theory we are interested in the asymptotic behaviour of the solution as $R\to\infty$. In this domain, by (2.1), the first term in (2.2) is a small quantity of higher order, so that $\mathbf{w}$ satisfies, to a first approximation, a linear equation, which we shall in fact first consider for the case without rotation, when $v_\varphi\equiv0$. The vector $\mathbf{v}_1$ is then Landau's solution. The solution of the linearized homogeneous system (2.2) will be sought as an expansion in eigenfunctions which satisfy (1.2):

$$\psi = \nu \sum_{n=2}^{\infty} y_n(x) R^{2-\alpha_n}, \quad w_R = -\nu \sum_{n=2}^{\infty} y_n'(x) R^{-\alpha_n} \tag{2.3}$$

$$w_\theta = -\frac{\nu}{\sqrt{1-x^2}} \sum_{n=2}^{\infty} (2-\alpha_n) y_n(x) R^{-\alpha_n}, \quad \frac{q}{\rho} = \nu^2 \sum_{n=2}^{\infty} g_n(x) R^{-\alpha_n-1}$$

Substitution of (2.3) in (2.2) leads to the equations

$$[(1-x^2) y_n'']' - 2\frac{1-x^2}{A-x} y_n'' + \Big[\alpha_n(\alpha_n - 5) + 2(\alpha_n+1)\frac{A^2-1}{(A-x)^2}\Big] y_n' - 4(\alpha_n-2)\Big[\frac{A^2-1}{(A-x)^3} - \frac{1}{A-x}\Big] y_n - (\alpha_n+1) g_n = 0 \tag{2.4}$$

$$g_n' + \alpha_n y_n'' - (\alpha_n - 2)\Big\{\frac{2}{A-x} y_n' + \Big[\frac{4x}{A-x} - 2\alpha_n \frac{A^2-1}{(A-x)^2} - \alpha_n(\alpha_n-3)\Big]\frac{y_n}{1-x^2}\Big\} = 0$$

Regular solutions of system (2.4) are only possible for certain values of α_n, connected with the eigenvalues of the operator of the system, which is a generalized Legendre operator. While such operators have good spectral properties /11/, the well-known spectral theorems cannot be used directly as applied to (2.4), because the parameter α_n appears essentially non-linearly in (2.4). We shall therefore confine ourselves to qualitative considerations and a numerical analysis. As a preliminary, we observe that the case $n=2$ plays an isolated role, since $\alpha_2 = 2$ for all A. This is connected with the law of conservation of flow rate (1.8), from which it follows at once that a term $\sim 1/R^2$ must be contained in expansion (2.3) for w_R. Formally, $\alpha_1 = 1$ is also an eigenvalue for all A; but it is superfluous, inasmuch as, by condition (2.1), the vector $\mathbf{w}$ must not contain terms $\sim 1/R$, as is reflected in the form of expansions (2.3).

In the case of zero Reynolds number $\mathrm{Re} = (J/\pi)^{1/2}/\nu$, when $A=\infty$, system (2.4) reduces to the equation

$$[(1-x^2) y_n'']'' + [(\alpha_n - 1)(\alpha_n - 2) + \alpha_n(\alpha_n+1)] y_n'' + \alpha_n(\alpha_n^2-1)(\alpha_n-2)\frac{y_n}{1-x^2} = 0 \tag{2.5}$$

If the regular solution of (2.5) is sought as a polynomial of degree $n+1$, an algebraic equation of the fourth degree is obtained for α_n, with the solutions $\alpha_n = n; n+2, -n+1, -n-1$. The condition of rest at infinity is obtained only when all the $\alpha_n > 0$ in expansions (2.3), so that the last two roots must be discarded; but the first two branches of eigenvalues are left. In the special case when $\mathrm{Re}=0$ these branches are superimposed, so that we can regard (2.5) as having an integer-valued double "spectrum" $\alpha_n = n$, while to each α_n there correspond two regular eigenfunctions

$$y_n^{(1)} = (1-x^2) P_{n-1}, \quad y_n^{(2)} = (1-x^2) Q_{n-3}, \quad n \geqslant 3 \tag{2.6}$$

where $P_n(x)$ and $Q_n(x)$ are polynomials of degree n.

Notice that, by (2.5), $y_n \sim (1-x^2)$ for $n \geqslant 3$, as is reflected in representation (2.6). With $\alpha_n = 2$, Eq.(2.5) together with the solution $x(1-x^2)$, has the eigenfunction $y_2 = x$, which is regular but does not satisfy the conditions $y_n(\pm 1) = 0$, and thus, by (1.8), ensures that the flow rate Q is non-zero. Notice that this does not contradict the condition of axial symmetry $w_\theta(R, \pm 1) = 0$, since the functions y_2 are not contained in expansion (2.3) for v_θ, inasmuch as $\alpha_2 = 2$.

The general solution of (2.5) contains two arbitrary constants

$$y_n = A_n y_n^{(1)} + B_n y_n^{(2)} \tag{2.7}$$

In our case $\mathrm{Re} \to 0$, problem (2.2) remains linear, not only as $R \to \infty$, but also for any R. The presence of the two infinite sequences $\{A_n\}, \{B_n\}$ means that in this case any two boundary conditions on the sphere $R = R_0$ can be satisfied; for instance, $v_R(R_0, x)$ and $v_\theta^-(R_0, x)$ can be arbitrarily specified from the space of continuous functions $C([-1, 1])$. This is possible by virtue of the well-known completeness of the set of polynomials in the space $C([-1, 1])$. Notice that the unusual situation involving the presence in the problem of two complete sets of eigenfunctions (2.6), which are linearly dependent, in fact ensures that two conditions can be satisfied.

As Re increases from zero, the branches $\alpha_n(\mathrm{Re})$ split up. while α_n themselves become non-integral for $n \geqslant 3$ (but $\alpha_2 = 2$). This is clear from Fig.1, where we show as continuous curves the results of a numerical solution of Eqs.(2.4) (curves of $\alpha_n(\mathrm{Re})$ for $n = 3, 4, 5$).

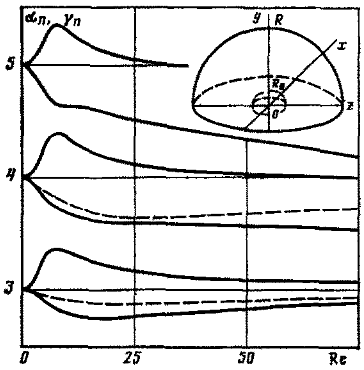

Fig.1

In essence, the splitting of the set of eigenvalues into two branches should not in principle change the situation: to each branch there must correspond a complete set of eigenfunctions (which are no longer polynomials). While this is easily seen physically, a strict proof of these properties encounters serious difficulties and has not yet been given.

3. By what has been said, for all Reynolds numbers the linearized equation (2.2) has a double spectral point, corresponding to $n = 2$ in expansions (2.3). In this case, by the general thoery, the system of basis functions must be augmented by associated eigenfunctions in order to be complete. An associated eigenfunction is a solution

$$w_R = -\nu\,[u'(x)\ln R + z'(x)]\frac{1}{R^2},\quad w_\theta = -\frac{\nu u(x)}{R^2\sqrt{1-x^2}} \qquad (3.1)$$

$$\frac{q}{\rho} = \frac{\nu^2}{R^3}\,[h(x)\ln R + g(x)]$$

It was shown in /12/ that a logarithmic resonance term must be introduced. The difference lies in the fact that here it must be present for all, and not just some, Reynolds numbers. Notice that the logarithmic term in the expression for v_R is obtained at once from the equation of continuity in (1.1), if a term $\sim 1/R^2$ is admitted in expansion (1.3) for v_θ.

Substituting (3.1) into (2.2) and integrating, we obtain the equations

$$Lu' = D = \text{const},\ h = -2u' + D/3 \qquad (3.2)$$

$$Lz' - 6z' = 3g - h - y_1'u' + y_1''u + u' + 2y_1u/(1-x^2) \qquad (3.3)$$

$$g' + 2z'' = u'' + \frac{2u - 2y_1'u - y_1u'}{1-x^2} - \frac{2xy_1u}{(1-x^2)^2}$$

The operator L is given by (1.7), and $y_1(x)$ by (1.4).

Eq.(3.2), like (1.7), has a regular solution only if $D = 0$. We then have

$$u(x) = B(1-x^2)\frac{1-Ax}{A(A-x)^2},\quad u' = BF_0(x) \qquad (3.4)$$

where $F_0(x)$ is given by (1.9). From (3.3), after integrating the second equation in the light of (1.4) and (3.4), we obtain the inhomogeneous equation

$$Lz' = f(x) \qquad (3.5)$$

$$f(x) = C + B\left[6 - \frac{6A^2-2}{A(A-x)} + \frac{3(A^2-1)}{(A-x)^2} - \frac{8A(A^2-1)}{(A-x)^3} + \frac{6(A^2-1)}{(A-x)^4}\right]$$

As in the case of problem (1.7), homogeneous Eq.(3.5) has the non-trivial solution (1.9), but now the right-hand side of (3.5) is such that, by a suitable choice of the constants B and C, we can ensure that the inhomogeneous problem is solvable. The second condition, connecting B and C, consists in specifying the flow rate Q.

Multiplying (3.5) by $(A-x)^2$, we obtain

$$[(1-x^2)(A-x)z'']' + 6(A^2-1)z' = (A-x)^2 f(x) \qquad (3.6)$$

Integration of (3.6) with respect to x from -1 to 1 gives the flow rate

$$Q = -\frac{2\pi\rho}{3(A^2-1)}\left\{C\left(A^2+\frac{1}{3}\right) + B\left[9A^2 - 5 + 4A(A^2-1)\ln\frac{A+1}{A-1}\right]\right\} \qquad (3.7)$$

From the condition for the selfadjoint Eq.(3.6) to be solvable:

$$\int_{-1}^{1} f(x)(A-x)^2 F_0(x)\,dx = 0,$$

we obtain

$$C\left[-4A^2 + \frac{20}{3} + 2\frac{(A^2-1)^2}{A}\ln\frac{A+1}{A-1}\right] + \qquad (3.8)$$

$$B\left[-44A^2 + \frac{196}{3} - \frac{8}{A^2} + \frac{(A^2-1)(22A^2-18)}{A}\ln\frac{A+1}{A-1}\right] = 0$$

The determinant of system (3.7) and (3.8) is non-zero for all $A \geqslant 1$, so that the constants

B and C are uniquely defined. Thus the general solution of Eq.(3.6) is

$$z'(x) = c_0 F_0(x) + F(x), \quad c_0 = \text{const} \tag{3.9}$$

where $F(x)$ is a regular particular solution of Eq.(3.6).

By using the method of varying the arbitrary constants, this solution can be written in quadratures

$$F(x) = F_0(x)\left\{\int_{-1}^{x}\left[\frac{\Phi(\xi)}{(\xi-x_1)^2(\xi-x_2)^2} - \frac{D_1}{(\xi-x_1)^2} - \frac{D_2}{(\xi-x_2)^2} - \frac{E_1}{\xi-x_1} - \frac{E_2}{\xi-x_2}\right]d\xi - \frac{D_1}{x-x_1} - \frac{D_2}{x-x_2} + E_1\ln|x-x_1| + E_2\ln|x-x_2|\right\}.$$

Here,

$$\Phi(\xi) = \frac{(A-\xi)^4}{(1-\xi^2)(\xi-x_3)^2}\int_{-1}^{\xi} F_0(\eta)(A-\eta)^2 f(\eta)\,d\eta$$

$$D_{1,2} = \frac{\Phi(x_{1,2})}{(x_1-x_2)^2}, \quad E_{1,2} = \frac{\Phi'(x_{1,2})}{(x_1-x_2)^2} \mp \frac{2\Phi(x_{1,2})}{(x_1-x_2)^3}.$$

From (1.9),

$$F_0(x) = -(x-x_1)(x-x_2)(x-x_3)(A-x)^{-3}$$

$$x_1 = A - 2\sqrt{A^2-1}\cos\frac{\alpha}{3}, \quad x_{2,3} = A + 2\sqrt{A^2-1}\cos\left(\frac{\alpha}{3} \pm \frac{\pi}{3}\right)$$

$$\cos\alpha = -\sqrt{A^2-1}/A, \quad x_{1,2} \in (-1, 1), \quad x_3 \,\overline{\in}\, [-1, 1].$$

This expression is regular in the closed interval $[-1, 1]$. Its form is due to additive isolation of the singularity, connected with inversion of the degenerate operator L.

4. Let us now consider the problem for any R and Re, when the non-linearity of Eq.(2.2) has to be taken into account. In the non-linear case, expansions (2.3), augmented by associated functions (3.1), are not a solution of (2.2), since, when (2.3) is substituted into (2.2), the non-linearity generates uncompensated terms with total degrees $1/R$. Hence expansion (2.3) is not closed.

When expanded in integral powers of $1/R$, the linear and non-linear terms in (2.2) will also be expressed in terms of certain integral powers, i.e., the integral powers have the group property. For the expansion to be closed, the family of non-integral powers must also have this property, so that the linear and non-linear terms give powers of this family. In view of these considerations, the closed expansion must be written in the form

$$\psi = \nu\left[\sum_{n=1}^{\infty} y_n(x) R^{2-\mu_n} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} u_{kn}(x)\frac{\ln^k R}{R^{2k-1}} R^{2-\mu_n}\right] \tag{4.1}$$

$$\mu_n = 1 + \sum_{j=2}^{\infty} n_j(\alpha_j - 1)$$

where n_j are non-negative integers, chosen in such a way that $\mu_n \leqslant \mu_{n+1}$. The functions $u_{kn}(x)$ satisfy linear inhomogeneous equations, which are obtained due to the non-linearity of (2.2). If, for instance, (3.1) is substituted into (2.2), the convective terms will generate inhomogeneous linear equations for the functions y_4, u_{14}, and u_{24}, i.e., inhomogeneous equations only make an appearance for terms of fairly high order.

Notice that (4.1) contains entirely as a subsequence the expansion (2.3), whose terms satisfy homogeneous equations; this would not happen if the α_n were integers. The expansion, which plays an interesting role in (4.1), may naturally be called, by analogy with the case of Laplace's equation, a generalized multipole expansion, which contains a double denumerable set of arbitrary constants A_n and B_n, which have to be found from the boundary conditions at $R = R_0$. Each multipole generates an entire sequence of terms of the expansion (4.1), while this sequence vanishes if there is no multipole. In particular, the dipole term generates in (4.1) a sequence of integral powers of $1/R$.

In short, the solution of our problem is a subset of a complete set of multipoles with the multipole sequences generated by them. In the double sum of (4.1) there only appears one arbitrary constant B, defined by Eqs.(3.7), (3.8), and proprotional to the flow rate Q. Hence the double sum may be termed a sequence whose characteristic feature is the presence of logarithmic terms.

With the aid of the above results, we can write explicitly the principal terms of the asymptotic expansion of the solution for large R. Up to terms of $o(R^{-2})$, we have

$$v_R = -\nu\left(\frac{y_1'}{R} + u'\frac{\ln R}{R^2} + \frac{z'}{R^2}\right), \quad v_\theta = -\frac{\nu}{\sqrt{1-x^2}}\left(\frac{y_1}{R} + \frac{u}{R^2}\right) \tag{4.2}$$

The functions y_1, u, and z' are given by (1.4), (3.4), and (3.9). They contain three arbitrary constants A, B, and c_0 (since c_0 and c_2 are arbitrary, the function $y_2' = c_2 F_0$ can be regarded as included in z'), which, in accordance with the usual methods of jet theory, have to be found in terms of the given integrals of conservation.

In all previous work, the two integrals of conservation (1.5) and (1.8) have been specified, i.e., the momentum J and flow rate Q; we pointed out above that these determine the constants A and B. But there is another exact integral of conservation for the Navier-Stokes equations, which may be specified independently of J and Q, and hence defines the constant c_0. This is the flux of the x-component of the angular momentum through the surface made up of a hemisphere of radius R and a ring in the $y = 0$ plane between circles of radii R and R_0 (see upper right of Fig.1).

$$L_x = -2\rho R^3 \int_0^\pi \left[v_R v_\theta - \nu\left(\frac{1}{R}\frac{\partial v_R}{\partial \theta} + \frac{\partial v_\theta}{\partial R} - \frac{v_\theta}{R}\right)\right]\sin\theta\, d\theta + \tag{4.3}$$
$$2\rho \int_{R_0}^{R} R^2\, dR \int_0^\pi \left[\frac{p}{\rho} + v_\varphi^2 - 2\nu\left(\frac{v_R}{R} + \frac{v_\theta \operatorname{ctg}\theta}{R}\right)\right]\cos\theta\, d\theta$$

If the integration were performed over the complete sphere, then $L_x = 0$ in the axisymmetric case. This is why the integral of conservation (4.3) has remained unnoticed in the literature.

In short, to describe jet non-similarity flow by means of the principal terms of asymptotic expansion (4.2), three integrals of conservation: J, Q, and L_x, have to be specified, and not just two.

This conclusion relates to the solution of the complete Navier-Stokes equations. Meanwhile, most work on jet theory uses the boundary-layer approximation. We shall first consider what happens with the exact solution in the situation when $\mathrm{Re} \to \infty$, $A \to 1$, so that, by (1.5), $J \to \infty$. The final result depends largely on how the flow rate Q varies on this passage to the limit, which in turn depends on the way in which Re increases. If the jet issues from a tube of radius a with characteristic fluid velocity v_0, then $J \sim v_0^2 a^2$, while $Q \sim v_0 a^2$, so that $Q \sim a\sqrt{J} \sim a(A-1)^{-1/2}$ and everyhing depends on the law of variation of $a(A)$. In particular, with $a = \mathrm{idem}$, i.e., a fixed tube geometry, the product $Q(A^2-1) \to 0$, so that system (3.7)-(3.8) reduces to the equations $C + 3B = 0$ and $C + 5B = 0$, which have the solution $C = B = 0$. Then, by (3.4), $u \equiv 0$, and the logarithmic terms in expansion (4.1) fall out. In this situation the expansion (1.3) is not contradictory, and for the asymptotic description of the non-similarity jet up to terms of order $1/R^2$, it suffices to specify only the two integrals of conservation J and L_x, while Q can be arbitrary.

Now consider the boundary-layer approximation. In cylindrical coordinates (r, φ, z) the boundary layer equations for an untwisted axisymmetric jet are

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{4.4}$$
$$\frac{\partial ru}{\partial z} + \frac{\partial rv}{\partial r} = 0, \quad \mathbf{v} = (v, 0, u)$$

Following /6/, we introduce the variable $\eta = r/(z\nu^{1/2})$ and write the stream function $\psi(\eta, z)$ as the expansion

$$\psi = \nu\left[a(\eta)z + a_0(\eta)z^{2-\beta_0} + \ldots + a_n(\eta)z^{2-\beta_n} + \ldots\right] \tag{4.5}$$

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{a'}{\eta}\frac{1}{z} + \frac{a_0'}{\eta}\frac{1}{z^{\beta_0}} + \ldots \tag{4.6}$$
$$v = -\frac{1}{r}\frac{\partial\psi}{\partial z} = \sqrt{\nu}\left\{\left(a' - \frac{a}{\eta}\right)\frac{1}{z} + \left[a_0' + (\beta_0 - 2)\frac{a_0}{\eta}\right]\frac{1}{z^{\beta_0}} + \ldots\right\}$$

The function $a(\eta)$ was obtained by Schlichting /1/

$$a(\eta) = \frac{\alpha^2\eta^2}{1 + \alpha^2\eta^2/4}, \quad \alpha = \sqrt{\frac{3J}{16\pi\rho\nu}} \tag{4.7}$$

where J is the jet momentum. Substituting (4.6) into (4.4) and repeating the arguments of Sect.2, we arrive at the linear spectral problem

$$\left(\frac{a_n}{\eta}\right) + \frac{1+a}{\eta}\left(\frac{a_n'}{\eta}\right)' + (1+\beta_n)\frac{a'}{\eta}\frac{a_n'}{\eta} - (\beta_n - 2)\left(\frac{a'}{\eta}\right)'\frac{a_n}{\eta} = 0 \tag{4.8}$$
$$a_n(0) = 0, \quad a_n'(0) = 0; \quad a_n/\eta \to 0, \quad \eta \to \infty$$

Notice that $\beta_0 = 2$ is an eigenvalue. The corresponding eigenfunction $a_0(\eta)$ was obtained by Loitsyanskii /6/. Following /6/, we introduce the variable $\xi = a/4$; then (4.8) becomes

$$\xi(1-\xi)^2 a_n''' + (1-4\xi)(1-\xi) a_n'' + 2\beta_n(1-\xi) a_n' + 4(\beta_n - 2) a_n = 0 \tag{4.9}$$

$$a_n(0) = 0; \quad a_n/\sqrt{1-\xi} \to 0, \quad \xi \to 1 \tag{4.10}$$

By comparison with (2.4), Eq.(4.9) has unusual spectral properties. First, given any β_n, there is a solution which is bounded at $\xi = 0; 1$,

$$S_2(\xi) = \xi^2 + \frac{2\beta_n - 5}{4 - \beta_n}\xi + \frac{\beta_n - 1}{\beta_n - 4} \tag{4.11}$$

so that there is a continuous spectrum. Moreover, for certain values of β_n, there are solutions in the form of polynomials $S_n(\xi)$ of degree n. The spectral values β_n satisfy the relation

$$(n-2)[n(n+3) - 2(\beta_n - 2)] = 0 \tag{4.12}$$

whence it is clear that, apart from $n = 2$, which corresponds to solution (4.11), there is a discrete integer-valued spectrum

$$\beta_n = n(n+3)/2 + 2, \ \beta_0 = 2, \ \beta_1 = 4, \ \beta_2 = 7, \ \beta_3 = 11, \ldots \tag{4.13}$$

which generates second non-trivial solutions of Eq.(4.9), bounded in an interval; these solutions, in combination with solutions (4.11), in general enable conditions (4.10) to be satisfied. However, with $\beta_n = 4; 7$, these solutions are the same as (4.11) and we must use non-analytic solutions, having the form, in the neighbourhood of $\xi = 1$,

$$a_n(\xi) = A_n + B_n(1-\xi)\ln(1-\xi) + \ldots \tag{4.14}$$

which likewise satisfy conditions (4.10). Thus, the spectrum of problem (4.9)-(4.10) is (4.13). The eigenfunctions corresponding to this spectrum form a system of multipoles of the boundary layer equations. As above, the multipoles generate multipole sequences, whose terms satisfy linear inhomogeneous equations.

Comparing β_n and α_n (the continuous curves of Fig.1), we see that the boundary layer approximation does not hold, starting with $\beta_n > 2$. Loitsyanskii's solution $\beta_0 = 2$ is the unique higher term of the expansion which is the same as the solution of the Navier-Stokes equations as $\mathrm{Re} \to \infty$ and for large z. All the higher approximations so far obtained in jet theory (up to the ninth) are terms of the same, namely, dipole sequence. Such approximations are clearly incorrect. This links up, in particular, with the paradoxical fact that the second approximation is in better agreement with experiment than is the third /9, 13/.

5. Consider a non-similarity twisted jet. The solution for the rotational velocity v_φ will be sought as

$$v_\varphi = \frac{\Gamma_1(x)}{R\sqrt{1-x^2}} + \frac{\Gamma_2(x)}{R^2\sqrt{1-x^2}} + \sum_{n=3}^{\infty} \frac{\Gamma_n(x)}{R^{\gamma_n}\sqrt{1-x^2}} + \ldots \tag{5.1}$$

The first two terms in (5.1) correspond to the well-known solutions /10, 14/, in which the integral powers are dictated by the laws of conservation of the x-component of the momentum /14/ and the z-component of the angular momentum /10/. A typical feature of the first term is the unboundedness of v_φ for $x = -1$, due to the impossibility of satisfying simultaneously the conditions $\Gamma_1(\pm 1) = 0$. This solution corresponds to the jet flow from a rapidly rotating tube, and in the limit, to a vertex thread. The solution for $y_1(x)$, which has minimal singularity, is regular for $x = 1$, and when rotation is present, is non-analytic: $y_1'(x) \sim \ln(1+x)$.

It is clear physically, however, that actual axisymmetric sources contain a semi-axis $x = -1$ inside the conducting tube, i.e., outside the domain of flow, so that non-analyticity at $x = -1$ is admissible; indeed, it was shown in /14, 15/ that it is precisely such solutions, characterized by a specified momentum of rotation, which correspond to actual sources of twisted jets, which cannot therefore be point sources. If we turn to a somewhat idealized statement of the problem with a given distribution of the velocity vector on a sphere of radius R_0, we find that, for the solution to be regular at $x = -1$, we have to put $\Gamma_1(x) \equiv 0$, with the result that Loitsyanskii's statement /6/ is obtained, in which the first term of the expansion is the dipole term, characterized by specifying the z-component of the flux of the angular momentum

$$L_z = -2\pi\rho R^3 \int_0^\pi \left[v_R v_\varphi - \nu\left(\frac{\partial v_\varphi}{\partial R} - \frac{v_\varphi}{R}\right)\right] \sin\theta \, d\theta$$

If (5.1) is substituted into the Navier-Stokes equation (1.1), and Landau's solution is used for v_R and v_θ, we arrive, as above, at a system of homogeneous linear equations, defining the eigenfunctions

$$(1-x^2)\Gamma_n'' - y_1\Gamma_n' + (\gamma_n - 1)(\gamma_n - y_1')\Gamma_n = 0 \tag{5.2}$$

Here, $y_1(x)$ is preserved in the form (1.4). The remaining equations will be inhomogeneous, which generates in (5.1) its own system of multiple sequences, which are not written down explicitly.

The eigenvalue $\gamma_2 = 2$. For $n > 2$ the γ_n are fractional and depend on the Reynolds number (the broken curves in Fig.1). With $\mathrm{Re} \to 0$ $(A \to \infty)$ $\gamma_n = n$, $\Gamma_n(x) = M_n(1-x^2)T_{n-2}(x)$ $(n \geqslant 2)$, where $T_n(x)$ are polynomials of degree n, and $M_n = \mathrm{const}$. The operator of Eq.(5.2) is a generalized Legendre operator, while the system of eigenfunctions $\Gamma_n(x)$ is complete in $L_2((-1, 1))$, so that $v_\varphi(R_0, x)$ can be any function of L_2. It must be said that, according to our computations, there are no eigenvalues γ_n in the interval (1.2), so that the dipole term $n=2$ is in fact the principal one in expansion (5.1), when $\Gamma_1(x) \equiv 0$.

Our solutions enable twisted jets with near-axial return currents to be described. Let the jet rotation be specified by the z-component L_z of the angular momentum. Then, using our integral (4.3) of conservation of L_x, we obtain the relation

$$L_x - a(\mathrm{Re})\,L_z^2 = b(\mathrm{Re})\,c_0 \tag{5.3}$$

in which we use the solutions (4.2); the constant c_0 is given in (3.9), while the velocity v_φ is found in /10/. Simple analytic working gives $L_x > 0$, $a(\mathrm{Re}) > 0$, and $b(\mathrm{Re}) > 0$. It is clear with the aid of (5.3) that, given the Reynolds number Re, by increasing the rotation, i.e., L_z, we can obtain negative values of c_0 as large as desired in absolute value. Then, by (4.2), for certain values of the spherical radius R, zones with $v_R < 0$ arise. However, as $\mathrm{Re} \to \infty$, the quantities $a \sim \mathrm{Re}^{-2}$, $b \sim \mathrm{Re}$. Hence it follows that, in the approximation of boundary layer theory $(a = 0)$, no return currents appear.

To sum up, the solution of the problem of a twisted jet, flowing from a "spherical source", is defined up to three denumerable sets of arbitrary constants $\{B_n\}$, $\{C_n\}$, and $\{M_n\}$, which enable any condition to be satisfied on the velocity vector, specified on a sphere of radius R_0. The principal terms of the asymptotic expansion are then determined by the four integrals of conservation: J, Q, L_x, L_z.

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REFERENCES

1. SCHLICHTING H., Laminare strahlausbreitung, Z. Angew. Math. und Mech., 13, 4,
2. LANDAU L.D., On a new exact solution of the Navier-Stokes equations, Dokl. Akad. Nauk SSSR, 43, 7, 1944.
3. RUMER YU.B., The problem of a submerged jet, PMM 16, 2, 1952.
4. VULIS L.A. and KASHKAROV V.P., The theory of a jet of viscous fluid (Teoriya strui vyazkoi zhidkosti), Nauka, Moscow, 1965.
5. KOROBKO V.I., Theory of non-similarity jets of viscous fluid (Teoriya neavtomodel'nykh strui vyzkoi zhidkosti), Izd-vo Sarat. un-ta, 1977.
6. LOITSYANSKII L.G., Propagation of a twisted jet in unbounded space submerged in the same fluid, PMM 19, 1, 1953.
7. DUBOV V.S., Propagation of a free twisted jet in submerged space, Tr. Leningr. politekhn. in-ta, 176, 1955.
8. FAL'KOVICH S.V., Propagation of a twisted jet in an unbounded space submerged in the same fluid, PMM 31, 2, 1967.
9. KOROBKO V.I. and FAL'KOVICH S.V., Development of twisted jet in unbounded space, Izv. Akad. Nauk SSSR, MZhG, 3, 1969.
10. TSUKKER M.S., A twisted jet propagating in a space submerged in the same fluid, PMM 19, 4, 1955.
11. TRIBEL' KH., Theory of interpolation, functional spaces, and differential operators /Russian translation/, Mir, Moscow, 1980.
12. GOL'DSHTIK M.A. and YAVORSKII N.I., The thermal problem for a submerged jet, PMM 48, 6, 1984.
13. AKHMEDOV R.B. (Ed.), Aerodynamics of a twisted jet (Aerodinamika zakruchennoi strui), Energya, Moscow, 1977.
14. GOL'DSHTIK M.A., On twisted jets, Izv. Akad. Nauk SSSR, MZhG, 1, 1979.
15. GOL'DSHTIK M.A., Vortical flows (Vikhrevye potoki), Nauka, Novosibirsk, 1981.

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